SOLUTION The velocity vector is $(e^t, -e^{-t}, -\sin t)$, which at $t = 1$ is the vector $(e, -1/e, -\sin 1)$. The particle is at $(e, 1/e, \cos 1)$ at $t = 1$. The equation of the tangent line is $\mathbf{l}(t) = (e, 1/e, \cos 1) + (t - 1)(e, -1/e, -\sin 1)$. At $t = 3$, the position on this line is

$$\begin{aligned}\mathbf{l}(3) &= \left(e, \frac{1}{e}, \cos 1\right) + 2\left(e, -\frac{1}{e}, -\sin 1\right) = \left(3e, -\frac{1}{e}, \cos 1 - 2\sin 1\right)\\ &\cong (8.155, -0.368, -1.143). \quad \blacktriangle\end{aligned}$$

EXERCISES

Sketch the curves that are the images of the paths in Exercises 1 to 4.

1. $x = \sin t$, $y = 4\cos t$, where $0 \leq t \leq 2\pi$

2. $x = 2\sin t$, $y = 4\cos t$, where $0 \leq t \leq 2\pi$

3. $\mathbf{c}(t) = (2t - 1, t + 2, t)$

4. $\mathbf{c}(t) = (-t, 2t, 1/t)$, where $1 \leq t \leq 3$

In Exercises 5 to 8, determine the velocity vector of the given path.

5. $\mathbf{c}(t) = 6t\mathbf{i} + 3t^2\mathbf{j} + t^3\mathbf{k}$

6. $\mathbf{c}(t) = (\sin 3t)\mathbf{i} + (\cos 3t)\mathbf{j} + 2t^{3/2}\mathbf{k}$

7. $\mathbf{r}(t) = (\cos^2 t, 3t - t^3, t)$

8. $\mathbf{r}(t) = (4e^t, 6t^4, \cos t)$

In Exercises 9 to 12, compute the tangent vector to the given path.

9. $\mathbf{c}(t) = (e^t, \cos t)$

10. $\mathbf{c}(t) = (3t^2, t^3)$

11. $\mathbf{c}(t) = (t \sin t, 4t)$

12. $\mathbf{c}(t) = (t^2, e^2)$

13. When is the velocity vector of a point on the rim of a rolling wheel *horizontal*? What is the speed at this point?

14. If the position of a particle in space is $(6t, 3t^2, t^3)$ at time t, what is its velocity vector at $t = 0$?

In Exercises 15 and 16, determine the equation of the tangent line to the given path at the specified value of t.

15. $(\sin 3t, \cos 3t, 2t^{5/2}); t = 1$

16. $(\cos^2 t, 3t - t^3, t); t = 0$

In Exercises 17 to 20, suppose that a particle following the given path $\mathbf{c}(t)$ *flies off on a tangent at* $t = t_0$*. Compute the position of the particle at the given time* t_1*.*

17. $\mathbf{c}(t) = (t^2, t^3 - 4t, 0)$, where $t_0 = 2, t_1 = 3$

18. $\mathbf{c}(t) = (e^t, e^{-t}, \cos t)$, where $t_0 = 1, t_1 = 2$

19. $\mathbf{c}(t) = (4e^t, 6t^4, \cos t)$, where $t_0 = 0, t_1 = 1$

20. $\mathbf{c}(t) = (\sin e^t, t, 4 - t^3)$, where $t_0 = 1, t_1 = 2$

2.5 Properties of the Derivative

In elementary calculus, we learn how to differentiate sums, products, quotients, and composite functions. We now generalize these ideas to functions of several variables, paying particular attention to the differentiation of composite functions. The rule for differentiating composites, called the *chain rule*, takes on a more profound form for functions of several variables than for those of one variable.

If f is a real-valued function of one variable, written as $z = f(y)$, and y is a function of x, written $y = g(x)$, then z becomes a function of x through substitution, namely, $z = f(g(x))$, and we have the familiar chain rule:

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = f'(g(x))g'(x).$$

If f is a real-valued function of three variables u, v, and w, written in the form $z = f(u, v, w)$, and the variables u, v, w are each functions of x, $u = g(x)$, $v = h(x)$, and $w = k(x)$, then by substituting $g(x)$, $h(x)$, and $k(x)$ for u, v, and w, we obtain z as a function of x: $z = f(g(x), h(x), k(x))$. The chain rule in this case reads:

$$\frac{dz}{dx} = \frac{\partial z}{\partial u}\frac{du}{dx} + \frac{\partial z}{\partial v}\frac{dv}{dx} + \frac{\partial z}{\partial w}\frac{dw}{dx}.$$

One of the goals of this section is to explain such formulas in detail.